Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE ASSOCIATED POINT OF SEVEN POINTS IN SPACE.*

By H. S. White.

From seven points in space an eighth point can be constructed to complete what Hesse and later geometricians have called a set of *associated* points. Unless their relative situation is in some way specialized, the construction is unique; and from any seven of the complete set the eighth is determined by the same method. That is, the eight points are symmetrically related. The most interesting properties of the set relate to surfaces of the second order, and it was natural that Hesse and many writers after him should employ quadrics in demonstrating even the uniqueness of the eighth point according to their several modes of construction. But the construction itself is linear,—by means of lines and planes exclusively. Accordingly the demonstration of uniqueness and symmetry does not actually require the use of quadrics.

It is proposed here to follow Hesse's† construction, to obtain an explicit equation for the eighth point as a covariant of the first seven in the set of associated points, and to prove from the algebraic forms its uniqueness and the symmetry of the set. Particularly interesting are equations 15 and 16.

1. **Geometric construction.** The first step in construction is to select one of the seven given points as a center of projection, or a first Brianchon point; and to adopt some definite order of sequence for the other six, regarding them as the vertices of a simple gauche hexagon. We shall use the numeral 7 for the first Brianchon point, and 123456 in cyclic order for the vertices of the skew hexagon. Next, draw three lines through point 7, each intersecting a pair of opposite sides of the hexagon. These are taken as diagonals of a first derived hexagon of Brianchon type, and their intersections with the sides of the given hexagon as vertices of this derived hexagon, inscribed in the first.

The original hexagon shall be called H, the first derived hexagon A, and a second derived hexagon B. Vertices of H are already named $1, 2, 3, \cdots, 6$; and its sides are properly indicated by $12, 23, \cdots, 56, 61$. Vertices of A may be denoted by a_{12}, a_{23}, etc., showing on what side of H they lie; while its sides are named α_1, α_2, etc., the side α_1 containing vertices a_{61} and a_{12}, α_2 connecting a_{12} and a_{23}, etc.

---

* Presented to the American Mathematical Society, New York, April 23, 1921.

† O. Hesse, "De curvis et superficiebus secundi ordinis," Crelle's Journal, vol. 20 (1840), pp. 285–308. For full references see Encyk. der math. Wissenschaften, vol. III2, pp. 248–9.

Construct in the third place six additional lines, which will be proved to form a second Brianchon hexagon inscribed in H. The first additional line, β_1, shall lie in the plane of sides 61 and 12 of H, and intersect the sides α_3 and α_5 of A. It meets also the side α_1, since it lies with α_1 in the plane 612. Similarly β_2 is to meet α_4 and α_6 and lie in the plane 123, and so forth.

It is to be proved that these six lines form a second Brianchon hexagon B_1 inscribed like the first in H. *The point where its three diagonals meet is the point* 8, *whose determination is the object of this construction.* Obviously points 7 and 8 are reciprocally related to the other six. It is to be proved also that point 8 is unchanged when points 7 and 6 exchange rôles; from this it follows that all eight points are symmetrically related.

2. **Formulæ for the 8th point.** The problem whose repeated solution yields the desired formula is this: given the equations of two points and a plane:

$$u_\alpha = 0, \qquad u_\beta = 0, \qquad a_x = 0,$$

to find the equation of the point where the plane meets the line joining the given points. It is of course

$$a_\alpha u_\beta - a_\beta u_\alpha = 0. \tag{1}$$

We may adopt certain abbreviations for formulæ. Point 1 shall be understood as having coördinates x_1^1, x_2^1, x_3^1, x_4^1. The equation $u_1x_1^1 + u_2x_2^1 + u_3x_3^1 + u_4x_4^1 = 0$ may be condensed to $ux^1 = 0$. The determinant of the coördinates of points 1, 2, 3, 4 is denoted by 1234, separated from other symbols by $+$, $-$, or when necessary.

For the order of points on the original hexagon H, adopt first 123456, and take point 7 as Brianchon point of the first derived hexagon A. The side α_1 is to join points

a_{61} where line 61 meets plane 734, and
a_{12} " " 12 " " 745.

These points have the equations

$$\begin{aligned} 7346\cdot ux^1 - 7341\cdot ux^6 &= 0, \\ 7451\cdot ux^2 - 7452\cdot ux^1 &= 0. \end{aligned} \tag{2}$$

Cyclic permutation gives the four other vertices of A. Write explicitly equations for the points a_{23}, a_{31} on α_3. Both α_1 and α_3 are to meet the plane 456, and their join-line in that plane is named β_5. *We wish to prove that* β_5 *intersects* β_6 *on the line* 56; that is, that the six lines β_1, β_2, etc., form a closed gauche hexagon B inscribed in H.

From equations (2), on the model of (1), we can write next the equation of the point where line α_1 meets plane 456.

$$\begin{aligned}&(7346\cdot 4561 - 7341\cdot 4566)(7451\cdot ux^2 - 7452\cdot ux^1)\\ &\qquad - (7451\cdot 4562 - 7452\cdot 4561)(7346\cdot ux^1 - 7341\cdot ux^6) = 0.\end{aligned}$$

The italicized determinant vanishes; and by the use of identities this equation becomes

$$\begin{aligned}(3)\quad &7346\cdot 4561(7451\cdot ux^2 - 7452\cdot ux^1)\\ &\qquad - 7456\cdot 4512(7346\cdot ux^1 - 7341\cdot ux^6) = 0.\end{aligned}$$

Similarly the equation of the point where line α_3 meets plane 456 is written thus:

$$\begin{aligned}(4)\quad &7614\cdot 4563\cdot(7562\cdot ux^3 - 7563\cdot ux^2)\\ &\qquad - 7564\cdot 2563\cdot(7614\cdot ux^3 - 7613\cdot ux^4) = 0.\end{aligned}$$

Both these points being in the plane 456, it is possible to express their equations linearly in ux^4, ux^5, and ux^6. Reduced by the aid of the usual identities equations (3) and (4) become respectively:

$$\begin{aligned}(3a)\quad &2561\cdot 7346\cdot 7451\cdot ux^4 + 4261\cdot 7346\cdot 7451\cdot ux^5\\ &\qquad + 4521\cdot 7345\cdot 7461\cdot ux^6 = 0.\end{aligned}$$

$$\begin{aligned}(4a)\quad &2563\cdot 7615\cdot 7643\cdot ux^4 - 2364\cdot 7635\cdot 7614\cdot ux^5\\ &\qquad + 2354\cdot 7635\cdot 7614\cdot vx^6 = 0.\end{aligned}$$

Any point on their join-line, β_5, has its equation compounded linearly of (3a) and (4a); and the point where β_5 intersects the line 56 has an equation containing only ux^5 and ux^6, hence ux^4 is to vanish in the combination. The point on β_5 and 56 is given therefore by the equation

$$\begin{aligned}(5)\quad &\left\{\begin{matrix}2563\cdot 7615\cdot 4261\cdot 7346\cdot 7451\\ - 2561\cdot 7451\cdot 2364\cdot 7635\cdot 7614\end{matrix}\right\} ux^5\\ &\qquad + \left\{\begin{matrix}2563\cdot 7615\cdot 4521\cdot 7345\cdot 7461\\ + 2561\cdot 7451\cdot 2354\cdot 7635\cdot 7614\end{matrix}\right\} ux^6 = 0.\end{aligned}$$

This equation can be written so as to exhibit better its invariance under a group of permutations.

$$\begin{aligned}(5a)\quad &7415\cdot ux^5\left\{\begin{matrix}6325\cdot 6347\cdot 6124\cdot 6157\\ - 6324\cdot 6357\cdot 6125\cdot 6147\end{matrix}\right\}\\ &\qquad - 7416\cdot ux^6\left\{\begin{matrix}5326\cdot 5347\cdot 5124\cdot 5167\\ - 5324\cdot 5367\cdot 5126\cdot 5147\end{matrix}\right\} = 0.\end{aligned}$$

Attend to the coefficients in braces. Each is of the form that we may call a *Pascalian.* Its vanishing is the condition that six points shall be projected from the seventh by rays of a cone of the second order. In the coefficient of ux^5, the point 6 is the center of projection; in that of ux^6, the point 5. It is known that when the center of projection is left unchanged, permutation of the other six points leaves a Pascalian invariant except as to sign. Odd permutations change the sign, even permutations do not. Accordingly *the point where line β_6 meets the side 56 is this same point on β_5.* For the points 4, 1 which appear in the factors 7415 and 7416 are adjacent to vertices 5 and 6 in the chosen order 123456; and have the same relation after the order is reversed: 432165. Also the two Pascalian coefficients are merely exchanged. Therefore the point may be designated indifferently by b_{56} or b_{65}, if it is remembered that 4 and 1 are adjacent to the pair of vertices 6 and 5.

A diagonal of hexagon B is completely described, without auxiliary memoranda, by naming the two opposite sides of hexagon H on which lie the two opposite vertices of B, e.g., $\beta(56, 23)$. It is unnecessary to mention the point 7 as different in function from 4 and 1, since the points 7, 4, 1 can be permuted among themselves without altering the position of the points b_{56} and b_{23}. For the Pascalians in equation (5*a*) are invariant under such permutation, and the external factors 7415, 7416 change sign simultaneously. Hence the notation $\beta(56, 23)$ or $\beta(65, 23)$, etc., cannot become ambiguous.

If this second derived hexagon, B, is of the Brianchon type, its three diagonals intersect. If that intersection is uniquely determined by the seven given points, *all diagonals of all second derived hexagons must intersect.* We shall prove that this is the case, and the uniqueness of the eighth point is thereby proved.

Equation (5*a*) shall be used as a model. To avoid any possible ambiguity we append all seven points in a definite order as index of a Pascalian; thus:

$$6325\cdot6347\cdot6124\cdot6157 - 6324\cdot6357\cdot6125\cdot6147 \equiv P_{6,\ 312547} \equiv -P_{6,\ 132547}.$$

Then equation (5*a*) can be rewritten:

$$\frac{7415\cdot ux^5}{P_{5,\ 312647}} - \frac{7416\cdot ux^6}{P_{6,\ 312547}} = 0. \tag{5b}$$

On this model, write equations of the points b_{12} and b_{34} as opposite vertices of a second derived hexagon.

$$\frac{7561\cdot ux^1}{P_{1,\ 234567}} - \frac{7562\cdot ux^2}{P_{2,\ 134567}} = 0. \tag{6}$$

$$\frac{7563\cdot ux^3}{P_{3,\ 234567}} - \frac{7564\cdot ux^4}{P_{4,\ 234567}} = 0. \tag{7}$$

If we add these equations, the point represented is certainly on the diagonal $\beta(12, 34)$. For it determines with 3 and 4 a plane containing the point b_{12}, and with 1 and 2 a plane containing the point b_{34}. Call this point $b(12, 34)$. Its equation is this:

$$\frac{7561 \cdot ux^1}{P_{1,\ 234567}} - \frac{7562 \cdot ux^2}{P_{2,\ 134567}} + \frac{7563 \cdot ux^3}{P_{3,\ 124567}} - \frac{7564 \cdot ux^4}{P_{4,\ 123567}} = 0. \tag{8}$$

This point coincides with $b(12, 35)$, whose equation is the following:

$$\frac{7461 \cdot ux^1}{P_{1,\ 235467}} - \frac{7462 \cdot ux^2}{P_{2,\ 135467}} + \frac{7463 \cdot ux^3}{P_{3,\ 125467}} - \frac{7465 \cdot ux^5}{P_{5,\ 123467}} = 0. \tag{9}$$

The identity of the two points is seen upon applying to (9) the relation

$$1234 \cdot ux^5 \equiv 1235 \cdot ux^4 + 1254 \cdot ux^3 + 1534 \cdot ux^2 + 5234 \cdot ux^1 \tag{10}$$

and consolidating the result by the aid of relations among three Pascalians like the following:

$$2345 \cdot 2367 \cdot P_{1,\ 234567} - 1345 \cdot 1367 \cdot P_{2,\ 134567} \equiv 1245 \cdot 1267 \cdot P_{3,\ 214567}. \tag{11}$$

For we have after the first substitution:

$$\begin{aligned} 0 = {} & \left(\frac{7461 \cdot 1234}{P_{1,\ 235467}} - \frac{7465 \cdot 5234}{P_{5,\ 123467}}\right) ux^1 \\ & - \left(\frac{7462 \cdot 1234}{P_{2,\ 135467}} + \frac{7465 \cdot 1534}{P_{5,\ 123467}}\right) ux^2 \\ & + \left(\frac{7463 \cdot 1234}{P_{3,\ 125467}} - \frac{7465 \cdot 1254}{P_{5,\ 123467}}\right) ux^3 - \frac{7465 \cdot 1235}{P_{5,\ 123467}} \cdot ux^4, \end{aligned}$$

and this becomes, after three applications of formulæ like (11),

$$\begin{aligned} 0 = {} & \frac{7165 \cdot 5231 \cdot P_{4,\ 235167}}{P_{1,\ 235467} \cdot P_{5,\ 123467}} \cdot ux^1 \\ & - \frac{7265 \cdot 1532 \cdot P_{4,\ 123567}}{P_{2,\ 135467} \cdot P_{5,\ 123467}} \cdot ux^2 \\ & + \frac{7365 \cdot 1253 \cdot P_{4,\ 123567}}{P_{3,\ 125467} \cdot P_{5,\ 123467}} \cdot ux^3 - \frac{7465 \cdot 1235}{P_{5,\ 123467}} \cdot ux^4. \end{aligned} \tag{12}$$

After removal of three factors, this is precisely equation (8). Therefore, as asserted above, the index of the diagonal can be changed by substitution of any one point, without disturbing the incidence of the line and the point $b(12, 34)$ given by equation (8). But in that manner in succession *all possible diagonals of second derived hexagons can be reached; therefore all contain this same* 8*th point,* whose uniqueness is thus proved.

In the construction of hexagons A and B is seen the reciprocal relation of their Brianchon points, the 7th and 8th of the associated set. The 7th is exchangeable with any of the first six; e.g., in equation (8) it is permutable with the 6th or 5th, quite obviously, hence also with the others. *Therefore all eight associated points are symmetrically related.*

3. **The extraneous factor.** Equation (8) represents a class of 35 equations, all equivalent geometrically since all represent equally the same eighth point of the set. In conciseness and symmetry it is not likely that this equation can be surpassed. *It contains, however, when cleared of fractions, the extraneous factor* 1234. For from the reduction in equation (12) we see that

$$1234\cdot H_{(12,\,35)} \equiv -\,1235\cdot H_{(12,\,34)} \tag{13}$$

where $H_{(12,\,34)} = 0$ is the left side of equation (8) cleared of fractions. Dividing out this factor 1234 would leave a form which does not change, save in sign (a skew contravariant), under any permutation of the first seven points:

$$\frac{H_{(12,\,34)}}{1234} \equiv -\frac{H_{(12,\,35)}}{1235}, \qquad \text{etc.} \tag{14}$$

From the structure of equation (8) it might be conjectured that we are dealing with a particular case of a form symmetric in two sets of coördinates, (u) and (v). Replace therefore the particular plane 756 by a parameter plane (v), and change signs of some terms by writing for index of P always some *cyclic* permutation of the order 1, 234567; extend the summation to all seven points.

$$\frac{vx^1\cdot ux^1}{P_{1,\,234567}} + \frac{vx^2\cdot ux^2}{P_{2,\,345671}} + \cdots + \frac{vx^7\cdot ux^7}{P_{7,\,123456}} = 0. \tag{15}$$

or briefly

$$\sum_{i=1}^{7}\left(\frac{ux^i\cdot vx^i}{P_i}\right) = 0 = S(u, v).$$

This includes as particular cases all 35 equivalent equations of type (8). It is, for every plane (v), the equation of the same point (x^8), since it may be compounded linearly from any four equations of type (8) which have not a common index-point, e.g., those which correspond to the four faces of a tetrahedron 4567. Further, this point $S(v, u) = 0$ is the polar of plane (v) with respect to the quadric envelope

$$S(u, u) = 0. \tag{16}$$

But because all planes have the same polar point (x^8), this quadric locus is the bundle, counted double, of all planes through that point.

$$S(u, u) = [ux^8]^2.$$

This squared equation has the merit of lacking the extraneous factors which occur in type (8), and is indeed of degree 14 in the coördinates of each of the seven given points, so that according to *Sturm* (Math. Annalen, 1) *it is free from all extraneous factors.*